

LOWER Q -HOMEOMORPHISMS WITH RESPECT TO p -MODULUS AND ORLICZ-SOBOLEV CLASSES

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Abstract

We show that under a condition of the Calderon type on φ the homeomorphisms f with finite distortion in $W_{\text{loc}}^{1,\varphi}$ and, in particular, $f \in W_{\text{loc}}^{1,s}$ for $s > n - 1$ are the so-called lower Q -homeomorphisms with respect to p -modulus where $Q(x)$ is equal to its outer p -dilatation $K_{p,f}(x)$.

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1 Introduction

In what follows, D is a domain in a finite-dimensional Euclidean space. Following Orlicz, see [26], given a convex increasing function $\varphi : [0, \infty) \rightarrow [0, \infty)$, $\varphi(0) = 0$, denote by L_φ the space of all functions $f : D \rightarrow \mathbb{R}$ such that

$$\int_D \varphi \left(\frac{|f(x)|}{\lambda} \right) dm(x) < \infty \quad (1.1)$$

for some $\lambda > 0$ where $dm(x)$ corresponds to the Lebesgue measure in D . L_φ is called the **Orlicz space**. If $\varphi(t) = t^p$, then we write also L_p . In other words, L_φ is the cone over the class of all functions $g : D \rightarrow \mathbb{R}$ such that

$$\int_D \varphi(|g(x)|) dm(x) < \infty \quad (1.2)$$

which is also called the **Orlicz class**, see [3].

The **Orlicz-Sobolev class** $W_{\text{loc}}^{1,\varphi}(D)$ is the class of locally integrable functions f given in D with the first distributional derivatives whose gradient ∇f belongs locally in D to the Orlicz class. Note that by definition $W_{\text{loc}}^{1,\varphi} \subseteq W_{\text{loc}}^{1,1}$. As usual, we write $f \in W_{\text{loc}}^{1,p}$ if $\varphi(t) = t^p$, $p \geq 1$. It is known that a continuous function f belongs to $W_{\text{loc}}^{1,p}$ if and only if $f \in ACL^p$, i.e., if f is locally absolutely continuous on a.e. straight line which is parallel to a coordinate axis, and if the first partial derivatives

of f are locally integrable with the power p , see, e.g., 1.1.3 in [24]. The concept of the distributional derivative was introduced by Sobolev [32] in \mathbb{R}^n , $n \geq 2$, and it is developed under wider settings at present, see, e.g., [28].

Later on, we also write $f \in W_{\text{loc}}^{1,\varphi}$ for a locally integrable vector-function $f = (f_1, \dots, f_m)$ of n real variables x_1, \dots, x_n if $f_i \in W_{\text{loc}}^{1,1}$ and

$$\int_D \varphi(|\nabla f(x)|) dm(x) < \infty \quad (1.3)$$

where $|\nabla f(x)| = \sqrt{\sum_{i,j} \left(\frac{\partial f_i}{\partial x_j}\right)^2}$. Note that in this paper we use the notation $W_{\text{loc}}^{1,\varphi}$ for more general functions φ than in the classical Orlicz classes giving up the condition on convexity of φ . Note also that the Orlicz–Sobolev classes are intensively studied in various aspects at present.

Recall that a homeomorphism f between domains D and D' in \mathbb{R}^n , $n \geq 2$, is called of **finite distortion** if $f \in W_{\text{loc}}^{1,1}$ and

$$\|f'(x)\|^n \leq K(x) \cdot J_f(x) \quad (1.4)$$

with a.e. finite function K where $\|f'(x)\|$ denotes the matrix norm of the Jacobian matrix f' of f at $x \in D$, $\|f'(x)\| = \sup_{h \in \mathbb{R}^n, |h|=1} |f'(x) \cdot h|$, and $J_f(x) = \det f'(x)$ is its Jacobian. We set $K_{p,f}(x) = \|f'(x)\|^p / J_f(x)$ if $J_f(x) \neq 0$, $K_{p,f}(x) = 1$ if $f'(x) = 0$ and $K_{p,f}(x) = \infty$ at the rest points.

First this notion was introduced on the plane for $f \in W_{\text{loc}}^{1,2}$ in the work [16]. Later on, this condition was changed by $f \in W_{\text{loc}}^{1,1}$ but with the additional condition $J_f \in L_{\text{loc}}^1$ in the monograph [15]. The theory of the mappings with finite distortion had many successors, see, e.g., a number of references in the monograph [23]. They had as predecessors the mappings with bounded distortion, see [27] and [34], in other words, the quasiregular mappings, see, e.g., [4], [5], [13], [21], [29] and [35].

Note that the above additional condition $J_f \in L_{\text{loc}}^1$ in the definition of the mappings with finite distortion can be omitted for homeomorphisms. Indeed, for each homeomorphism f between domains D and D' in \mathbb{R}^n with the first partial derivatives a.e. in D , there is a set E of the Lebesgue measure zero such that f satisfies (N) -property by Lusin on $D \setminus E$ and

$$\int_A J_f(x) dm(x) = |f(A)| \quad (1.5)$$

for every Borel set $A \subset D \setminus E$, see, e.g., 3.1.4, 3.1.8 and 3.2.5 in [8]. On this base, it is easy by the Hölder inequality to verify, in particular, that if $f \in W_{\text{loc}}^{1,1}$ is a homeomorphism and $K_f \in L_{\text{loc}}^q$ for some $q > n - 1$, then also $f \in W_{\text{loc}}^{1,p}$ for some $p > n - 1$, that we use further to obtain corollaries.

In this paper $H^k(A)$, $k \geq 0$, $\dim_H A$ denote the **k-dimensional Hausdorff measure** and the **Hausdorff dimension**, correspondingly, of a set A in \mathbb{R}^n , $n \geq 1$. It was shown in [11] that a set A with $\dim_H A = p$ can be transformed into a set $B = f(A)$ with $\dim_H B = q$ for each pair of numbers p and $q \in (0, n)$ under a quasiconformal mapping f of \mathbb{R}^n onto itself, cf. also [1] and [2].

2 Preliminaries

First of all, the following fine property of functions f in the Sobolev classes $W_{\text{loc}}^{1,p}$ was proved in the monograph [12], Theorem 5.5, and can be extended to the Orlicz-Sobolev classes. The statement follows directly from the Fubini theorem and the known characterization of functions in Sobolev's class $W_{\text{loc}}^{1,1}$ in terms of ACL (absolute continuity on lines), see, e.g., Section 1.1.3 in [24].

Theorem 2.1. *Let Ω be an open set in \mathbb{R}^n , $n \geq 3$, and let $f : \Omega \rightarrow \mathbb{R}^n$ be a continuous open mapping in the class $W_{\text{loc}}^{1,\varphi}(\Omega)$ where $\varphi : [0, \infty) \rightarrow [0, \infty)$ is increasing with the condition*

$$\int_1^\infty \left[\frac{t}{\varphi(t)} \right]^{\frac{1}{n-2}} dt < \infty. \quad (2.1)$$

. Then f has a total differential a.e. in Ω .

Corollary 2.1. *If $f : \Omega \rightarrow \mathbb{R}^n$ is a homeomorphism in $W_{\text{loc}}^{1,1}$ with $K_f \in L_{\text{loc}}^p$ for $p > n - 1$, then f is differentiable a.e.*

Theorem 2.2. *Let U be an open set in \mathbb{R}^n , $n \geq 3$, and let $\varphi : [0, \infty) \rightarrow [0, \infty)$ is increasing with the condition (2.1). Then each continuous mapping $f : U \rightarrow \mathbb{R}^m$, $m \geq 1$, in the class $W_{\text{loc}}^{1,\varphi}$ has the (N)-property (furthermore, it is locally absolutely continuous) with respect to the $(n-1)$ -dimensional Hausdorff measure on a.e. hyperplane \mathcal{P} which is parallel to a fixed coordinate hyperplane \mathcal{P}_0 . Moreover, $H^{n-1}(f(E)) = 0$ whenever $|\nabla f| = 0$ on $E \subset \mathcal{P}$ for a.e. such \mathcal{P} .*

Note that, if the condition (2.1) holds for an increasing function φ , then the function $\varphi_* = \varphi(ct)$ for $c > 0$ also satisfies (2.1). Moreover, the Hausdorff measures are quasi-invariant under quasi-isometries. By the Lindelöf property of \mathbb{R}^n , $U \setminus \{x_0\}$ can be covered by a countable collection of open segments of spherical rings in $U \setminus \{x_0\}$ centered at x_0 and each such segment can be mapped onto a rectangular oriented segment of \mathbb{R}^n by some quasi-isometry, see, e.g., I.5.XI in [20] for the Lindelöf theorem. Thus, applying piecewise Theorem 2.2, we obtain the following.

Corollary 2.2. *Under (2.1) each $f \in W_{\text{loc}}^{1,\varphi}$ has the (N)-property (furthermore, it is locally absolutely continuous) on a.e. sphere S centered at a prescribed point $x_0 \in \mathbb{R}^n$. Moreover, $H^{n-1}(f(E)) = 0$ whenever $|\nabla f| = 0$ on $E \subseteq S$ for a.e. such sphere S .*

3 Moduli of families of surfaces

The recent development of the moduli method in the connection with modern classes of mappings can be found in the monograph [23] and further references therein.

Let ω be an open set in $\overline{\mathbb{R}^k}$, $k = 1, \dots, n-1$. A (continuous) mapping $S : \omega \rightarrow \mathbb{R}^n$ is called a k -dimensional surface S in \mathbb{R}^n . Sometimes we call the image $S(\omega) \subseteq \mathbb{R}^n$ the surface S , too. The number of preimages

$$N(S, y) = \text{card } S^{-1}(y) = \text{card } \{x \in \omega : S(x) = y\}, \quad y \in \mathbb{R}^n \quad (3.1)$$

is said to be a **multiplicity function** of the surface S . In other words, $N(S, y)$ denotes the multiplicity of covering of the point y by the surface S . It is known that the multiplicity function is lower semicontinuous, i.e.,

$$N(S, y) \geq \liminf_{m \rightarrow \infty} N(S, y_m)$$

for every sequence $y_m \in \mathbb{R}^n$, $m = 1, 2, \dots$, such that $y_m \rightarrow y \in \mathbb{R}^n$ as $m \rightarrow \infty$, see e.g. [?], p. 160. Thus, the function $N(S, y)$ is Borel measurable and hence measurable with respect to every Hausdorff measure H^k ; see e.g. [31], p. 52.

Recall that a k -dimensional Hausdorff area in \mathbb{R}^n (or simply **area**) associated with a surface $S : \omega \rightarrow \mathbb{R}^n$ is given by

$$\mathcal{A}_S(B) = \mathcal{A}_S^k(B) := \int_B N(S, y) dH^k y \quad (3.2)$$

for every Borel set $B \subseteq \mathbb{R}^n$ and, more generally, for an arbitrary set that is measurable with respect to H^k in \mathbb{R}^n , cf. 3.2.1 in [8]. The surface S is called **rectifiable** if $\mathcal{A}_S(\mathbb{R}^n) < \infty$, see 9.2 in [23].

If $\varrho : \mathbb{R}^n \rightarrow [0, \infty]$ is a Borel function, then its **integral over** S is defined by the equality

$$\int_S \varrho d\mathcal{A} := \int_{\mathbb{R}^n} \varrho(y) N(S, y) dH^k y. \quad (3.3)$$

Given a family Γ of k -dimensional surfaces S , a Borel function $\varrho : \mathbb{R}^n \rightarrow [0, \infty]$ is called **admissible** for Γ , abbr. $\varrho \in \text{adm } \Gamma$, if

$$\int_S \varrho^k d\mathcal{A} \geq 1 \quad (3.4)$$

for every $S \in \Gamma$. Given $p \in (0, \infty)$, the **p -modulus** of Γ is the quantity

$$M_p(\Gamma) = \inf_{\varrho \in \text{adm } \Gamma} \int_{\mathbb{R}^n} \varrho^p(x) dm(x). \quad (3.5)$$

We also set

$$M(\Gamma) = M_n(\Gamma) \quad (3.6)$$

and call the quantity $M(\Gamma)$ the **modulus of the family** Γ . The modulus is itself an outer measure in the space of all k -dimensional surfaces.

We say that Γ_2 is **minorized** by Γ_1 and write $\Gamma_2 > \Gamma_1$ if every $S \subset \Gamma_2$ has a subsurface that belongs to Γ_1 . It is known that $M_p(\Gamma_1) \geq M_p(\Gamma_2)$, see [?], p. 176-178. We also say that a property P holds for **p -a.e.** (almost every) k -dimensional surface S in a family Γ if a subfamily of all surfaces of Γ , for which P fails, has the p -modulus zero. If $0 < q < p$, then P also holds for q -a.e. S , see Theorem 3 in [?]. In the case $p = n$, we write simply a.e.

Remark 3.1. The definition of the modulus immediately implies that, for every $p \in (0, \infty)$ and $k = 1, \dots, n - 1$

- (1) p -a.e. k -dimensional surface in \mathbb{R}^n is rectifiable,
- (2) given a Borel set B in \mathbb{R}^n of (Lebesgue) measure zero,

$$\mathcal{A}_S(B) = 0 \tag{3.7}$$

for p -a.e. k -dimensional surface S in \mathbb{R}^n .

The following lemma was first proved in [17], see also Lemma 9.1 in [23].

Lemma 3.1. *Let $k = 1, \dots, n - 1$, $p \in [k, \infty)$, and let C be an open cube in \mathbb{R}^n , $n \geq 2$, whose edges are parallel to coordinate axis. If a property P holds for p -a.e. k -dimensional surface S in C , then P also holds for a.e. k -dimensional plane in C that is parallel to a k -dimensional coordinate plane H .*

The latter a.e. is related to the Lebesgue measure in the corresponding $(n - k)$ -dimensional coordinate plane H^\perp that is perpendicular to H .

The following statement, see Theorem 2.11 in [18] or Theorem 9.1 in [23], is an analog of the Fubini theorem, cf. e.g. [31], p. 77. It extends Theorem 33.1 in [?], cf. also Theorem 3 in [?], Lemma 2.13 in [?] and Lemma 8.1 in [23].

Theorem 3.1. *Let $k = 1, \dots, n - 1$, $p \in [k, \infty)$, and let E be a subset in an open set $\Omega \subset \mathbb{R}^n$, $n \geq 2$. Then E is measurable by Lebesgue in \mathbb{R}^n if and only if E is measurable with respect to area on p -a.e. k -dimensional surface S in Ω . Moreover, $|E| = 0$ if and only if*

$$\mathcal{A}_S(E) = 0 \tag{3.8}$$

on p -a.e. k -dimensional surface S in Ω .

Remark 3.2. Say by the Lusin theorem, see e.g. Section 2.3.5 in [8], for every measurable function $\varrho : \mathbb{R}^n \rightarrow [0, \infty]$, there is a Borel function $\varrho^* : \mathbb{R}^n \rightarrow [0, \infty]$ such that $\varrho^* = \varrho$ a.e. in \mathbb{R}^n . Thus, by Theorem 3.1, ϱ is measurable on p -a.e. k -dimensional surface S in \mathbb{R}^n for every $p \in (0, \infty)$ and $k = 1, \dots, n - 1$.

We say that a Lebesgue measurable function $\varrho : \mathbb{R}^n \rightarrow [0, \infty]$ is **p -extensively admissible** for a family Γ of k -dimensional surfaces S in \mathbb{R}^n , abbr. $\varrho \in \text{ext}_p \text{adm } \Gamma$, if

$$\int_S \varrho^k d\mathcal{A} \geq 1 \quad (3.9)$$

for p -a.e. $S \in \Gamma$. The **p -extensive modulus** $\overline{M}_p(\Gamma)$ of Γ is the quantity

$$\overline{M}_p(\Gamma) = \inf_{\mathbb{R}^n} \int \varrho^p(x) dm(x) \quad (3.10)$$

where the infimum is taken over all $\varrho \in \text{ext}_p \text{adm } \Gamma$. In the case $p = n$, we use the notations $\overline{M}(\Gamma)$ and $\varrho \in \text{ext adm } \Gamma$, respectively. For every $p \in (0, \infty)$, $k = 1, \dots, n-1$, and every family Γ of k -dimensional surfaces in \mathbb{R}^n ,

$$\overline{M}_p(\Gamma) = M_p(\Gamma). \quad (3.11)$$

4 Ring Q -homeomorphisms and their properties

Recall some necessary notions. Let $E, F \subseteq \mathbb{R}^n$ be arbitrary domains. Denote by $\Delta(E, F, G)$ the family of all curves $\gamma : [a, b] \rightarrow \mathbb{R}^n$, which join E and F in G , i.e. $\gamma(a) \in E, \gamma(b) \in F$ and $\gamma(t) \in G$ for $a < t < b$. Set $d_0 = \text{dist}(x_0, \partial G)$ and let $Q : G \rightarrow [0, \infty]$ be a Lebesgue measurable function. Denote

$$A(x_0, r_1, r_2) = \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\},$$

and

$$S_i = S(x_0, r_i) = \{x \in \mathbb{R}^n : |x - x_0| = r_i\}, \quad i = 1, 2. \quad (4.1)$$

We say that a homeomorphism $f : G \rightarrow \mathbb{R}^n$ is *the ring Q -homeomorphism with respect to p -module at the point $x_0 \in G$* , ($1 < p \leq n$) if the inequality

$$\mathcal{M}_p(\Delta(f(S_1), f(S_2), f(G))) \leq \int_A Q(x) \cdot \eta^p(|x - x_0|) dx \quad (4.2)$$

is fulfilled for any ring $A = A(x_0, r_1, r_2)$, $0 < r_1 < r_2 < d_0$ and for every measurable function $\eta : (r_1, r_2) \rightarrow [0, \infty]$, satisfying

$$\int_{r_1}^{r_2} \eta(r) dr \geq 1. \quad (4.3)$$

The homeomorphism $f : G \rightarrow \mathbb{R}^n$ is the *ring Q -homeomorphism with respect to p -module in the domain G* , if inequality (4.2) holds for all points $x_0 \in G$. The properties of the ring Q -homeomorphisms for $p = n$ are studied in [?].

The ring Q -homeomorphisms are defined in fact locally and contain as a proper subclass of Q -homeomorphisms (see [?]). A necessary and sufficient condition for homeomorphisms to be ring Q -homeomorphisms with respect to p -module at a point given in [?], asserts:

Proposition 4.1. *Let G be a bounded domain in \mathbb{R}^n , $n \geq 2$ and let $Q : G \rightarrow [0, \infty]$ belong to L^1_{loc} . A homeomorphism $f : G \rightarrow \mathbb{R}^n$ is a ring Q -homeomorphism with respect to p -module at $x_0 \in G$ if and only if for any $0 < r_1 < r_2 < d_0 = \text{dist}(x_0, \partial G)$,*

$$\mathcal{M}_p(\Delta(f(S_1), f(S_2), f(G))) \leq \frac{\omega_{n-1}}{I^{p-1}},$$

where S_1 and S_2 are the spheres defined in (4.1)

$$I = I(x_0, r_1, r_2) = \int_{r_1}^{r_2} \frac{dr}{r^{\frac{n-1}{p-1}} q_{x_0}^{\frac{1}{p-1}}(r)},$$

and $q_{x_0}(r)$ is the mean value of Q over $|x - x_0| = r$. Note that the infimum in the right-hand side of (4.2) over all admissible η satisfying (4.3) is attained only for the function

$$\eta_0(r) = \frac{1}{I r^{\frac{n-1}{p-1}} q_{x_0}^{\frac{1}{p-1}}(r)}.$$

In this sections we establish the relationship between the ring and lower Q -homeomorphisms with respect to p -module.

Theorem 4.1. *Every lower Q -homeomorphism with respect to p -module $f : G \rightarrow G^*$ at $x_0 \in G$, with $p > n - 1$ and $Q \in L^{\frac{n-1}{p-n+1}}_{\text{loc}}$, is a ring \tilde{Q} -homeomorphism with respect to α -module at x_0 with $\tilde{Q} = Q^{\frac{n-1}{p-n+1}}$ and $\alpha = \frac{p}{p-n+1}$.*

5 Lower Q -homeomorphisms and Orlicz-Sobolev classes

Let D and D' be two bounded domains in \mathbb{R}^n , $n \geq 2$ and $x_0 \in D$. Given a Lebesgue measurable function $Q : D \rightarrow [0, \infty]$, a homeomorphism $f : D \rightarrow D'$ is called the *lower Q -homeomorphism with respect to p -modulus at x_0* if

$$\mathcal{M}_p(f(\Sigma_\varepsilon)) \geq \inf_{\rho \in \text{ext}_{\text{padm}} \Sigma_\varepsilon} \int_{A_\varepsilon(x_0)} \frac{\rho^p(x)}{Q(x)} dm(x), \quad (5.1)$$

where

$$A_\varepsilon(x_0) = \{x \in \mathbb{R}^n : \varepsilon < |x - x_0| < \varepsilon_0\}, \quad 0 < \varepsilon < \varepsilon_0, \quad 0 < \varepsilon_0 < \sup_{x \in G} |x - x_0|,$$

and Σ_ε denotes the family of all spheres centered at x_0 of radii r , $\varepsilon < r < \varepsilon_0$, located in D .

Theorem 5.1. *Let D and D' be domains in \mathbb{R}^n , $n \geq 3$, and let $\varphi : [0, \infty) \rightarrow [0, \infty)$ be increasing with the condition (2.1). Then each homeomorphism $f : D \rightarrow D'$ of finite distortion in the class $W_{\text{loc}}^{1,\varphi}$ is a lower Q -homeomorphism at every point $x_0 \in D$ with $Q(x) = K_{p,f}(x)$.*

Proof. Let B be a (Borel) set of all points $x \in D$ where f has a total differential $f'(x)$ and $J_f(x) \neq 0$. Then, applying Kirszbraun's theorem and uniqueness of approximate differential, see, e.g., 2.10.43 and 3.1.2 in [8], we see that B is the union of a countable collection of Borel sets B_l , $l = 1, 2, \dots$, such that $f_l = f|_{B_l}$ are bi-Lipschitz homeomorphisms, see, 3.2.2, 3.1.4 and 3.1.8 in [8]. With no loss of generality, we may assume that the B_l are mutually disjoint. Denote also by B_* the set of all points $x \in D$ where f has the total differential but with $f'(x) = 0$.

By the construction the set $B_0 := D \setminus (B \cup B_*)$ has Lebesgue measure zero, see Theorem 2.1. Hence by Theorem 2.4 in [18] or by Theorem 9.1 in [23] the area $\mathcal{A}_S(B_0) = 0$ for a.e. hypersurface S in \mathbb{R}^n and, in particular, for a.e. sphere $S_r := S(x_0, r)$ centered at a prescribed point $x_0 \in \overline{D}$. Thus, by Corollary 2.2 $\mathcal{A}_{S_r^*}(f(B_0)) = 0$ as well as $\mathcal{A}_{S_r^*}(f(B_*)) = 0$ for a.e. S_r where $S_r^* = f(S_r)$.

Let Γ be the family of all intersections of the spheres S_r , $r \in (\varepsilon, \varepsilon_0)$, $\varepsilon_0 < d_0 = \sup_{x \in D} |x - x_0|$, with the domain D . Given $\varrho_* \in \text{adm } f(\Gamma)$, $\varrho_* \equiv 0$ outside of $f(D)$, set $\varrho \equiv 0$ outside of D and on B_0 ,

$$\varrho(x) : = \varrho_*(f(x)) \|f'(x)\| \quad \text{for } x \in D \setminus B_0.$$

Arguing piecewise on B_l , $l = 1, 2, \dots$, we have by 1.7.6 and 3.2.2 in [8] that

$$\int_{S_r} \varrho^{n-1} d\mathcal{A} \geq \int_{S_r^*} \varrho_*^{n-1} d\mathcal{A} \geq 1$$

for a.e. S_r and, thus, $\varrho \in \text{ext}_p \text{adm } \Gamma$.

The change of variables on each B_l , $l = 1, 2, \dots$, see, e.g., Theorem 3.2.5 in [8], and countable additivity of integrals give the estimate

$$\int_D \frac{\varrho^p(x)}{K_{p,f}(x)} dm(x) \leq \int_{f(D)} \varrho_*^p(x) dm(x)$$

and the proof is complete.

Corollary 5.1. *Each homeomorphism f of finite distortion in \mathbb{R}^n , $n \geq 3$, in the class $W_{\text{loc}}^{1,s}$ for $s > n - 1$ is a lower Q -homeomorphism at every point $x_0 \in D$ with $Q(x) = K_{p,f}(x)$.*

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